Calculus of Variations and Infinite Series

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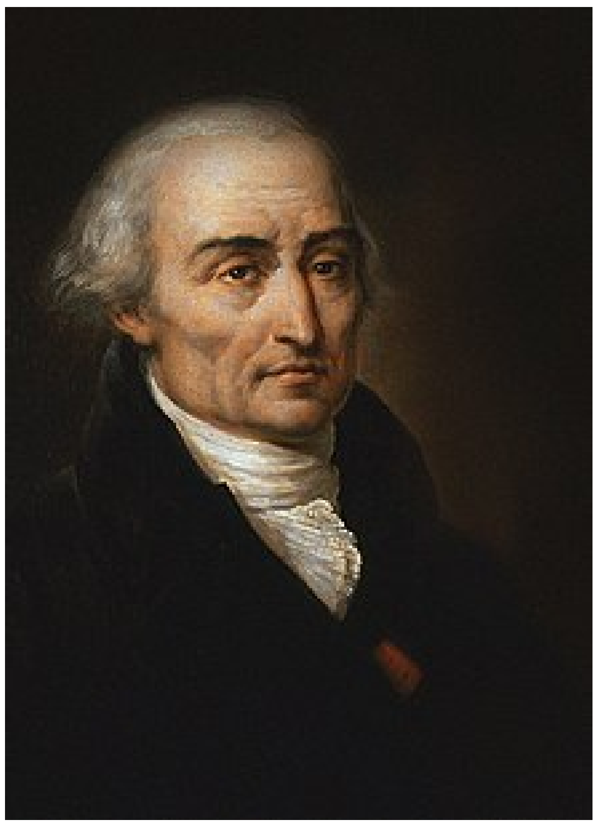

Contents

0 Introduction

0.1 Preface

The module this course belongs to is assessed over two short, open-book examinations. As a consequence, these notes are very brief and contain very little proof or reasoning.

1 Calculus of Variations

1.1 Functionals

Definition (Functional)**.** A functional maps a function or set of functions to the real numbers,

$$S : C^n(\mathbb{R}) \to \mathbb{R} \tag{1}$$

$$S : f(x) \to S[f] = \int_a^b F[f, f', x]\mathrm{d}x, \tag{2}$$

where $C^n(\mathbb{R})$ is the space of at least n-differentiable functions on $\mathbb{R}$ and F is an expression involving f, f', x.

A physically interesting example of a functional is distance D, which is a function of the path γ,

$$\gamma = (x(t), y(t))\,. \tag{3}$$

The total distance is found by integrating along the path,

$$D[\gamma] = \int_A^B \mathrm{d}s = \int_{t_A}^{t_B} \frac{\mathrm{d}s}{\mathrm{d}t}\mathrm{d}t = \int_{t_A}^{t_B} \sqrt{\dot{x}^2 + \dot{y}^2}\mathrm{d}t. \tag{4}$$

Definition (Functional Derivative)**.** Consider some functional S which is a function of f, f', x, where we can write

$$S[f] = \int_a^b L[f, f', x]\mathrm{d}x. \tag{5}$$

Under a small, arbitrary perturbation δf where $\delta f(a) = \delta f(b) = 0$,

$$\delta S = \int_a^b \delta L \mathrm{d}x = \int_a^b \left(\frac{\partial L}{\partial f}\delta f + \frac{\partial L}{\partial f'}\delta f' \right) \mathrm{d}x. \tag{6}$$

Integrating by parts, and applying the boundary conditions, we find that the functional derivative is

$$\frac{\delta S}{\delta f} = \frac{\partial L}{\partial f} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial L}{\partial f'} \tag{7}$$

where

$$\delta S = \int_a^b \mathrm{d}x \delta f \left(\frac{\delta S}{\delta f} \right). \tag{8}$$

When the functional derivative is zero, the change in the functional is zero and hence the functional is extremised.

1.2 Euler-Lagrange Equations

Consider a functional

$$S[f] = \int_a^b L[f, f'.x]\mathrm{d}x. \tag{9}$$

The stationary paths of S are found by solving the Euler-Lagrange equations,

$$\frac{\partial L}{\partial f} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial L}{\partial f'}\right) = 0. \tag{10}$$

The integrand L of the functional is called the Lagrangian. It typically has the form

$$L = T - V, \tag{11}$$

where T and V denote the kinetic and potential energy, respectively.

Consider the total derivative of the Lagrangian,

$$\frac{\mathrm{d}L}{\mathrm{d}x} = \frac{\partial L}{\partial x} + \frac{\partial L}{\partial y}y' + \frac{\partial L}{\partial y'}y''. \tag{12}$$

If $\dfrac{\partial L}{\partial x} = 0$, then substituting the Euler-Lagrange equation yields

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(L - y'\frac{\partial L}{\partial y'}\right) = 0, \tag{13}$$

implying the quantity in the bracket is a conserved quantity in x.

1.3 Lagrange Multipliers

Lagrange Multipliers provide a way to solve constrained optimisation problems. Consider a function $f(\mathbf{x}) : \mathbb{R}^n \to \mathbb{R}$ and a constraint $p(\mathbf{x}) = 0$; we want to find the value of $\mathbf{x}$ that maximises (or minimises) f while satisfying the constraint. To maximise f, we need

$$\nabla f \cdot \mathrm{d}\mathbf{x} = 0, \tag{14}$$

where $\mathrm{d}\mathbf{x}$ is parallel to p. Hence, if we solve

$$\nabla f = \lambda \nabla p \quad \text{where} \quad p = 0, \tag{15}$$

we solve our problem. We have reformulated our constrained optimisation problem as an unconstrained optimisation problem, where we ask for the stationary points of a new function $\phi(\mathbf{x}, \lambda)$,

$$\phi(\mathbf{x}, \lambda) = f(\mathbf{x}) - \lambda p(\mathbf{x}). \tag{16}$$

Maximising against $\mathbf{x}$ obtains the $\nabla f = \lambda \nabla p$ condition, and maximising against λ gives the condition $p = 0$.

1.3.1 Lagrange Multipliers Recipe

An easy recipe to extremise a function $f(x, y) : \mathbb{R} \to \mathbb{R}$ subject to a constraint $C(x, y) = 0$:

1. Find the Euler-Lagrange equations for $f - \lambda C$
2. Manipulate the E.L.s to find x and y as functions of λ
3. Substitute into constraint equation and solve for λ
4. Substitute λ into E.L.s to find solutions for x and y.

Of course, this recipe can be extended for a function of many variables.

1.4 Generalised Coordinates and Degrees of Freedom

Definition (Configuration Space)**.** The configuration space is a vector space containing generalised coordinates $\xi(t)$ that specify the configuration of the system. Generalised coordinates capture all information about the system in one single vector. In general, for a system of N different free particles, the configuration space has $3N$ dimensions (degrees of freedom). Although the Cartesian coordinates are the standard choice, generalised coordinates can take many forms - often, it is much more natural to work with different coordinates and calculations can be simplified significantly.

2 Infinite Series

2.1 Series and Summation

Definition (Partial Sum)**.** Let $\{a_n : n \in \mathbb{N}\}$ be a sequence of terms, then the partial sum S_n is defined as the finite sum

$$S_n = \sum_{i=1}^{n} a_i. \tag{17}$$

If $\lim_{n\to\infty} S_n$ exists and is finite, we say the series converges to S. Otherwise, it diverges.

Definition (Convergence)**.** Formally,

$$\forall\ \epsilon > 0\ \exists\ n_0 \in \mathbb{N}, S \in \mathbb{R} \text{ s.t. } |S_n - S| < \epsilon\ \forall\ n \geq n_0, \tag{18}$$

i.e. for all ϵ greater than 0 there exists some positive integer n_0 and some real number S such that the absolute difference between S_n and S is less than epsilon for all n greater than or equal to n_0.

Definition (Absolute convergence)**.** A series $\sum a_n$ converges absolutely if the series $\sum |a_n|$ converges. It can be proven that

$$\sum a_n \text{ converges absolutely } \implies \sum a_n \text{ converges.} \tag{19}$$

Definition (Unconditional convergence)**.** A series $\sum a_n$ converges unconditionally if the series $\sum a_{\pi(n)}$ converges absolutely for every bijection $\pi : \mathbb{N} \to \mathbb{N}$, i.e. no matter how we order the elements of a_n, the sum still converges. It can be proven that a series converges absolutely if an only if it converges unconditionally.

2.2 Convergence Tests

It is important to determine if a series converges. Here are a few popular convergence tests for infinite series with terms a_n.

Lemma (Ratio test)**.** If

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| \text{ exists and } < 1, \tag{20}$$

then $S = \sum a_n$ converges. If

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| \text{ exists and } > 1, \tag{21}$$

then $S = \sum a_n$ diverges. If the limit is equal to 1 then the test is inconclusive.

Lemma (Comparision test)**.** Let (a_n) and (b_n) be non-negative sequences and suppose that there exists some C and n_0 such that $\forall\ a_n \leq Cb_n$ and $n \geq n_0$, then, if $\sum b_n$ converges, $\sum a_n$ converges. Similarly, if $\sum a_n$ diverges, $\sum b_n$ diverges.

Lemma (Integral test)**.** If $f(x)$ is a continuous, monotonically decreasing function for $x \geq x_0 \geq n_0$ and $f(n) \geq a_n$, then

$$\int_{x_0}^{\infty} f(x)\mathrm{d}x < \infty \implies \sum_{n=0}^{\infty} a_n \text{ converges.} \tag{22}$$

Lemma (Cauchy's nth rooth test)**.** If $a_n \geq 0\ \forall\ n \in \mathbb{N}$,

1. $\lim_{n\to\infty} a_n^{1/n} < 1 \implies \sum a_n$ converges
2. $\lim_{n\to\infty} a_n^{1/n} > 1 \implies \sum a_n$ diverges
3. $\lim_{n\to\infty} a_n^{1/n} = 1 \implies \sum a_n$ inconclusive

Lemma (Condensation test)**.** If $a_n > 0$ and $a_{n+1} < a_n\ \forall\ n \in \mathbb{N}$ (i.e. series is monotonically decreasing), then

$$\sum_{n=1}^{\infty} a_n \text{ converges} \iff \sum_{k=0}^{\infty} 2^k a_{2^k} \text{ converges} \tag{23}$$

Lemma (Alternating series test)**.** Suppose $u_n \in \mathbb{R}$ and $u_n \geq 0$. Suppose also that u_n forms a decreasing sequence with $u_n \to 0$ as $n \to \infty$. Then, the alternating series

$$\sum_{n=1}^{\infty} (-1)^{n+1} u_n \text{ converges} \tag{24}$$

2.3 Integration tricks

For whatever reason, this course made a short segway into integration tricks. Most of these are quite simple, but here is a quick recap.

2.3.1 Differentiate under integral

For some integral I, multiply the integrand by some function of α, for example $e^{-\alpha x}$, to yield a new integral $I(\alpha)$ where $I = I(0)$. We can then find the derivative $\mathrm{d}I/\mathrm{d}\alpha$ and rearrange into a more favourable form. Finally, we integrate with respect to α to find $I = I(0)$.

2.3.2 Polar Coordinates

Sometimes, changing into a more natural set of coordinates can make integration easier. Polar coordinates are sometimes a good choice.

$$x = r\cos\theta \tag{25}$$

$$y = r\sin\theta \tag{26}$$

$$\mathrm{d}A = \mathrm{d}x\mathrm{d}y = r\mathrm{d}r\mathrm{d}\theta \tag{27}$$

2.3.3 Laplace's method (steepest descent approximation)

Suppose $f(x)$ is a twice continuously differentiable function on $[a, b]$, and there exists a unique point $x_0 \in (a, b)$ such that

$$f(x_0) = \max_{x\in[a,b]} f(x) \quad \text{and} \quad f''(x_0) < 0, \tag{28}$$

then

$$\int_a^b \mathrm{e}^{Mf(x)}\mathrm{d}x \approx \mathrm{e}^{Mf(x_0)}\sqrt{\frac{2\pi}{M|f''(x_0)|}}. \tag{29}$$

This is exact in the limit $M \to \infty$.

2.4 Gamma function

Definition (Gamma function)**.** The gamma function is defined as

$$\Gamma(x) = \int_0^\infty \mathrm{d}t t^{x-1}e^{-t}. \tag{30}$$

For $x > 0$. For a positive integer $n \in \mathbb{N}$,

$$\Gamma(n) = (n-1)!. \tag{31}$$

For a positive half integer $n/2$, $n \in \mathbb{N}$,

$$\Gamma\left(\frac{n}{2}\right) = \sqrt{\pi}\frac{(n-2)!!}{2^{(n-1)/2}}, \tag{32}$$

with $\Gamma(1/2) = \sqrt{\pi}$. Integrating Γ by parts yields a useful relation,

$$\Gamma(x+1) = x\Gamma(x). \tag{33}$$

We can use this relation to define Γ for negative arguments. For example,

$$\Gamma(-1/2) = \Gamma(1/2-1) = \frac{1}{1/2-1}\Gamma(1/2) = -2\sqrt{\pi}. \tag{34}$$

2.5 Beta function

Definition (Beta function)**.** The beta function $B(p,q)$ is defined by

$$B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}. \tag{35}$$

It can also be represented as an integral in several forms:

- Polynomial form

$$B(p,q) = \int_0^1 t^{p-1}(1-t)^{q-1}\mathrm{d}t, \tag{36}$$

- Trigonometric form

$$\frac{1}{2}B\left(\frac{1}{2}(p+1), \frac{1}{2}(q+1)\right) = \int_0^{\pi/2} \sin^n(t)\cos^m(t)\mathrm{d}t, \tag{37}$$

- Alternative polynomial form

$$B(p,q) = \int_0^{\infty} \frac{t^{p-1}}{(1+t)^{p+q}}\mathrm{d}t, \tag{38}$$

where $\Re(p), \Re(q) > 0$.

Series Convergence/Divergence Flow Chart